* Eigen Values and Eigen Vectors *

:- Let $A = [a_{ij}]$ be an $n \times n$ matrix. A non-zero vector 'x' is said to be "characteristic vector" of 'A' if $\exists$ a scalar 'λ' $\ni$ $\boxed{Ax = \lambda x}$

If $Ax = \lambda x$, $(x \neq 0)$ we say that 'x' is eigen vector (or) characteristic vector of 'A' corresponding to the eigen value (or) characteristic value 'λ' of 'A'.

eg:- Take $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then

$$\Rightarrow Ax = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1x$$

$\therefore$ $\boxed{Ax = 1.x}$

$\therefore$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of 'A' corresponding to the eigen value '1' of 'A'.

eg:- Consider $x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$\Rightarrow Ax = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0x$$

$\therefore$ $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is an eigen vector of 'A' corresponding to the eigen value '0' of 'A'.

Note: An eigen value of a square matrix 'A' can be zero. But a zero vector cannot be an eigen vector of 'A'.

To find the eigen vectors of a matrix :- Let $A = [a_{ij}]$ be a $n \times n$ matrix. Let 'x' be an eigen vector of 'A' corresponding to the eigen value 'λ'

then by definition, $AX = \lambda X$

i.e, $AX = \lambda \cdot IX$

$\Rightarrow AX - \lambda \cdot IX = 0$

$\Rightarrow (A - \lambda I) X = 0$

this is a homogenous system of 'n'- equations in 'n' unknowns.

This will have a non-zero solution X, $\Leftrightarrow |A - \lambda I| = 0$

-* $(A-\lambda I)$ is called "characteristic matrix" of 'A'. Also $|A-\lambda I|$ is a polynomia n 'λ' of degree 'n' and is called the "characteristic polynomial" of 'A'. lso $|A-\lambda I| = 0$ is called the "characteristic equation" of 'A'.

solving this equation, we get the roots $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$ of he characteristic Eq^n. these are the characteristic roots (or) eigen values the matrix.

Corresponding to each one of these 'n' eigen values, we can find ne characteristic vector 'X'. Consider the homogenous system

$$(A - \lambda_i I) X_i = 0 \text{ for } i = 1, 2, 3, \ldots, n$$

the non-zero sol^n 'X_i' of this system is the eigen vector of 'A' corresponding to the eigen value 'λ_i'.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a given matrix.

Its characteristic matrix is $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$

$|A-\lambda I|$ by expansion is a polynomial $\phi(\lambda)$ of degree 'n'. is called the "characteristic polynomial" of 'A'.

The characteristic eqⁿ of 'A' is

$$\phi(\lambda) = |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its roots. These are the characteristic roots of 'A'. These are also referred to as "eigen values" (or) "Latent roots" (or) "Proper values" of 'A'.

Consider each one of these eigen values say 'λ'. The eigen vector 'X' corresponding to the eigen value 'λ' is obtained by solving the homogenous system.

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

& determining the

Problems:

① Find the characteristic roots of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solⁿ: Given matrix 'A' is $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Its characteristic matrix is $A-\lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

the characteristic Eqn of 'A' is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)\left[(3-\lambda)(2-\lambda)-2\right] - 2(2-\lambda-1) + 1(2-3+\lambda) = 0$$

$$\Rightarrow (2-\lambda)\left[6-3\lambda-2\lambda+\lambda^2-2\right] - 2(-\lambda+1) + (\lambda-1) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-5\lambda+4) + 2\lambda - 2 + \lambda - 1 = 0$$

$$\Rightarrow 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda - 3 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \quad \lambda = 1 \;\Big|\; \begin{array}{cccc} 1 & -7 & 11 & -5 \\ 0 & 1 & -6 & 5 \\ \hline 1 & -6 & 5 & 0 \end{array}$$

$$\Rightarrow (\lambda-1)(\lambda^2-6\lambda+5) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-5\lambda-\lambda+5) = 0$$

$$\Rightarrow (\lambda-1)\left[\lambda(\lambda-5)-(\lambda-5)\right] = 0$$

$$\Rightarrow (\lambda-1)(\lambda-5)(\lambda-1) = 0$$

$$\Rightarrow \boxed{\lambda = 5 \;\Big|\; \lambda = 1, 1}$$

∴ the characteristic roots of 'A' are 5, 1, 1

2). Find the eigen values and the corresponding eigen vectors of $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Sol:- Given $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

the characteristic matrix is $A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$

the characteristic Eqn of 'A' is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0 \Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda - \lambda + 6 = 0$$

$$\Rightarrow \lambda(\lambda-6) - (\lambda-6) = 0$$

$$\Rightarrow \boxed{\lambda = 6, 1}$$

∴ the characteristic roots of 'A' are 6, 1.

Consider the system $\begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad [\because (A-\lambda I)X = 0]$

Eigen vector corresponding to the eigen value $\lambda = 1$:-

$$\rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the system of equation is

$4x_1 + 4x_2 = 0$

$x_1 + x_2 = 0 \rightarrow x_2 = -x_1$,

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ where } \alpha \neq 0 \text{ is a scalar.}$$

put $\boxed{x_1 = \alpha}$, $\boxed{x_2 = -\alpha}$

Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is eigen vector of 'A' corresponding to the eigen value 'λ'

Eigen vector corresponding to the eigen value $\lambda = 6$:-

$$\rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the system of equation is $-x_1 + 4x_2 = 0$

$x_1 - 4x_2 = 0 \rightarrow x_1 = 4x_2$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Put $\boxed{x_1 = 4\alpha}$

$\downarrow$

$\boxed{x_2 = \alpha}$

Hence $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is eigen vector of 'A', corresponding to the eigen value $\lambda = 6$

(3) * Find the eigen values and the corresponding eigen vectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol^n :- the characteristic Eq^n of 'A' is $|A - \lambda I| = 0$ $\rightarrow \left| \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$

i.e.)

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\rightarrow (-2-\lambda)\left[(1-\lambda)(-\lambda) - 12\right] - 2(-2\lambda - 6) - 3(-4 + (1-\lambda)) = 0$

$\rightarrow (-2-\lambda)\left[-\lambda + \lambda^2 - 12\right] + 2(2\lambda + 6) - 3(-3-\lambda) = 0$

$\rightarrow (-2-\lambda)(\lambda^2 - \lambda - 12) + 4\lambda + 12 + 9 + 3\lambda = 0 \Rightarrow -2\lambda^2 + 2\lambda + 24 - \lambda^3 + \lambda^2 + 12\lambda + 4\lambda + 3\lambda + 12 + 9 = 0$

$\rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0 \Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\Rightarrow \lambda^3+\lambda^2-21\lambda-45=0$ $\quad$ $\lambda=-3$
$$\begin{array}{c|cccc} & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$(-27+9-45+63-9)$

$72-72=$

$\Rightarrow (\lambda+3)(\lambda^2-2\lambda-15)=0 \Rightarrow (\lambda+3)(\lambda^2-5\lambda+3\lambda-15)=0 \Rightarrow (\lambda+3)[\lambda(\lambda-5)+3(\lambda-5)]=$

$\Rightarrow (\lambda+3)(\lambda-5)(\lambda+3)=0$

$\therefore \lambda=5, \lambda=-3,-3.$

Eigen vector of 'A' corresponding to $\lambda=-3$:–

If 'x' is an eigen vector of 'A' corresponding to the eigen value of 'A', we have $(A-\lambda I)x=0$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda=-3$:–
$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The ~~augmented~~ matrix of the system is $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$ $R_2 \to R_2-2R_1$, $R_3 \to R_3+R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

Hence, we have

$x_1+2x_2-3x_3=0 \Rightarrow x_1 = 3x_3-2x_2$

put $\boxed{x_2=\alpha}$, $\boxed{x_3=\beta}$

$\boxed{x_1 = 3\beta-2\alpha}$

Hence
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2\alpha+3\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$
is eigen vector corresponding to the eigen value $\lambda=-3$.

Eigen vector of 'A' corresponding to $\lambda=5$:– we have $(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the Augmented matrix of the System is

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow -R_3$.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$; $R_3 = R_3 + 7R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = \dfrac{R_2}{-8}$; $R_3 = \dfrac{R_3}{16}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + 5x_3 = 0$; $x_2 + 2x_3 = 0$; $\boxed{x_3 = \alpha}$

$x_1 = -5\alpha + 4\alpha$ $\qquad$ $x_2 = -2x_3$

$\boxed{x_1 = -\alpha}$ $\qquad$ $\boxed{x_2 = -2\alpha}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

//.

Note : (i). Sum of the eigen values of 'A' is same as the trace of 'A'.

(ii). The product of the eigen values of 'A' is same as the determinant of 'A'.

1) * Verify that the sum of eigen values is equal to the trace of 'A' for the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the corresponding eigen vectors.

Sol:- The characteristic equation of 'A' is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)\left[(5-\lambda)(3-\lambda) - 1\right] + (\lambda - 3 + 1) + (1 - 5 + \lambda) =$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

$$\begin{array}{c|cccc} \lambda = 3 & 1 & -11 & 36 & -36 \\ & 0 & 3 & -24 & 36 \\ \hline & 1 & -8 & 12 & \boxed{0} \end{array}$$

(by Horner's method)

$\Rightarrow (\lambda - 3)(\lambda^2 - 8\lambda + 12) = 0$

$\Rightarrow \lambda - 3 = 0$

$\boxed{\lambda = 3}$

$\lambda^2 - 8\lambda + 12 = 0$

$(\lambda - 6)(\lambda - 2) = 0$

$\boxed{\lambda = 6}$; $\boxed{\lambda = 2}$

$\therefore \lambda = 2, 3, 6.$

Sum of the eigen values $= 2 + 3 + 6 = 11$

Trace of $A = 3+5+3 = 11$ (9)

$\therefore$ the sum of the eigen values = trace of 'A' is verified.

<u>Eigen vectors corresponding to $\lambda = 3$</u> :—

Consider $(A - \lambda I)x = 0$.

$$\Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$.

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$;

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 = 0$; $x_2 - x_3 = 0$; $\boxed{x_3 = \alpha}$

$\boxed{x_1 = \alpha}$ $\boxed{x_2 = \alpha}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 2$:— Consider $(A - \lambda I)X = 0$.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$; $R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put

$\Rightarrow x_1 - x_2 + x_3 = 0$; $2x_2 = 0$; $\boxed{x_3 = \alpha}$

$\boxed{x_1 = -\alpha}$ $\quad$ $\boxed{x_2 = 0}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 6$:—

Consider $(A - \lambda I)X = 0$.

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$.

$$\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -1 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$; $R_3 = R_3 + 3R_1$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$

$-4 - -2 \times 2$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 - 3x_3 = 0$

$-2\alpha - \cancel{x_2} - 3\cancel{x_3} = 0 \Rightarrow \boxed{x_3 = -5\alpha}$

$-2x_1 - 4x_3 = 0$

$-2x_1 = 4x_3$

$\boxed{x_1 = -2\alpha}$

$\boxed{x_2 =}$

(5). Find the Eigen values and corresponding to the Eigen vectors of the matrix
$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(6). Find the Eigen values and Eigen vectors of the following matrices.

(i). $\begin{bmatrix} -2 & 5 \\ -1 & 4 \end{bmatrix}$

(ii). $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

(iii). $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$

(iv). $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(v). $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

) Find the sum and product of the eigen values of the matrix.

(i). $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

AW (ii). $A = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 6 \\ 2 & -2 & 3 \end{bmatrix}$

Sol (i). Sum of the eigen values = trace of the matrix

$= 2+4+2 = 8.$

Product of the eigenvalues $= \det A = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix}$

$= 16.$

(8). Prove that zero is eigen value of a matrix $\Leftrightarrow$ it is singular.

Sol:- Let 'A' be a square matrix of order 'n'.

Let $\lambda = 0$ be the eigen value of 'A'. Then $|A - \lambda I| = 0$.

$\Rightarrow |A| = 0.$

$\Rightarrow$ 'A' is singular matrix.

Conversely, suppose that 'A' is singular matrix.

i.e, $|A| = 0$

$\Rightarrow |A - 0(I)| = 0.$

$\Rightarrow$ '0' is the eigen value of 'A'.

$\therefore$ Zero is the eigen value of a matrix $\Leftrightarrow$ it is singular.

Find the Eigen values and the corresponding Eigen vectors of the matrix.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

sol: the characteristic equation of 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \rightarrow (2-\lambda)[(5-\lambda)(3-\lambda)] - 2[2(3-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)(15-5\lambda-3\lambda+\lambda^2) - 4(3-\lambda) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-8\lambda+15) - 12 + 4\lambda = 0$$

$$\Rightarrow 2\lambda^2 - 16\lambda + 30 - \lambda^3 + 8\lambda^2 - 15\lambda + 4\lambda - 12 = 0$$

$$\Rightarrow -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0$$

$$\Rightarrow \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$$

$$\begin{array}{c|cccc} \lambda=1 & 1 & -10 & 27 & -18 \\ & 0 & 1 & -9 & 18 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 9\lambda + 18) = 0$$

$$\rightarrow (\lambda-1)(\lambda^2 - 6\lambda - 3\lambda + 18) = 0 \Rightarrow (\lambda-1)[\lambda(\lambda-6) - 3(\lambda-6)] = 0$$

$$\Rightarrow (\lambda-1)(\lambda-6)(\lambda-3) = 0$$

$$\boxed{\lambda = 1, 3, 6}$$

∴ the eigen values of 'A' are 1, 3, 6.

the eigen vector corresponding to eigen value $\lambda = 1$:-

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$x_1 + 2x_2 = 0$

$2x_1 + 4x_2 = 0 \Rightarrow x_1 + 2x_2 = 0 \Rightarrow \boxed{x_1 = -2x_2}$

$2x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

Put $\boxed{x_2 = k}$

$\boxed{x_1 = -2k}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

the eigen vector corresponding to eigen value $\lambda = 3$:-

$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-x_1 + 2x_2 + \cancel{0} = 0 \Rightarrow \boxed{x_1 = 2x_2} \Rightarrow \boxed{x_1 = 0}$

$2x_1 + 2x_2 = 0 \Rightarrow \boxed{x_1 = -x_2} \Rightarrow 2x_2 + x_2 = 0$

$\Rightarrow \boxed{x_2 = 0}$

Put $\boxed{x_3 = k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

the eigen vector corresponding to $\lambda = 3$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The eigen vector corresponding to eigen value $\lambda = 6$:–

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$-4x_1 + 2x_2 = 0 \rightarrow \boxed{2x_1 = x_2} \Rightarrow \boxed{x_2 = 2k}$

$2x_1 - x_2 = 0 \rightarrow 2x_1 - 2x_1 = 0$ Put $\boxed{x_1 = k}$

$-3x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ the eigen vector corresponding to eigen value $\lambda = 6$ is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

2(A)

* (a). $\begin{bmatrix} -2 & 5 \\ -1 & 4 \end{bmatrix}$ (b). $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ find eigen values and find corresponding eigen vectors.

Solⁿ:– (a). The characteristic eqⁿ is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 5 \\ -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)(4-\lambda) + 5 = 0$$

$\Rightarrow -8 + 2\lambda - 4\lambda + \lambda^2 + 5 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow \lambda^2 - 3\lambda + \lambda - 3 = 0 \Rightarrow \lambda(\lambda - 3) + (\lambda - 3) = 0$

$\lambda = 3, -1.$

$\lambda = 3$:– $\begin{bmatrix} -5 & 5 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + 5x_2 = 0$

$-x_1 + x_2 = 0$

$\boxed{x_1 = x_2}$

Put $\boxed{x_1 = k} \Rightarrow \boxed{x_2 = k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$:– $\begin{bmatrix} -1 & 5 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -x_1 + 5x_2 = 0$

$-x_1 + 5x_2 = 0$

$\boxed{x_1 = 5x_2}$

Put $\boxed{x_2 = k \\ x_1 = 5k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5k \\ k \end{bmatrix} = k\begin{bmatrix} 5 \\ 1 \end{bmatrix},$$

(b). $\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (8-\lambda)(2-\lambda) + 8 = 0$

$\Rightarrow 16 - 8\lambda - 2\lambda + \lambda^2 + 8 = 0$

$\Rightarrow \lambda^2 - 10\lambda + 24 = 0$

$\Rightarrow \lambda^2 - 6\lambda - 4\lambda + 24 = 0$

$\Rightarrow \lambda(\lambda - 6) - 4(\lambda - 6) = 0$

$(\lambda - 6)(\lambda - 4) = 0$

$\Rightarrow \lambda = 6, 4$

$\lambda = 6$:– $\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 2x_1 - 4x_2 = 0$

$\Rightarrow 2x_1 = 4x_2$

$\Rightarrow \boxed{x_1 = 2x_2}$

Put $\boxed{x_2 = k}$

$\boxed{x_1 = 2k}$ $\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 4$:– $\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1 - 4x_2 = 0$

$4x_1 = 4x_2$

$\boxed{x_1 = x_2}$

$2x_1 - 2x_2 = 0$

$\Rightarrow \boxed{x_1 = x_2}$ Put $x_2 = k$ then $x_1 = k$

$\therefore \boxed{x_1 = x_2 = k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Multiplying bothsides of (2) by 'A', we get

$\Rightarrow A(k_1x_1+k_2x_2) = A(0) = 0$

$\Rightarrow k_1(Ax_1)+k_2(Ax_2) = 0$

$\Rightarrow k_1(\lambda_1x_1)+k_2(\lambda_2x_2) = 0$ ——— (3) $[\because (1)]$

(3) − λ_2(2) gives $\Rightarrow k_1(\lambda_1x_1)+k_2(\lambda_2\cancel{x_2}) - k_1x_1\lambda_2 - \lambda_2k_2\cancel{x_2} = 0$

$\Rightarrow k_1(\lambda_1-\lambda_2)x_1 = 0$

$\Rightarrow k_1 = 0$ $(\because x_1 \neq 0 \ \& \ \lambda_1 \neq \lambda_2)$

$k_2 = 0$

But this contradicts our assumption that k_1, k_2 are not zeros.

Hence our assumption that 'x_1' and 'x_2' are linearly dependent is wrong.

Hence the two eigen vectors corresponding to the two different eigen values are linearly independent (L.I).

Hence Proved.

D: 11/7/2013

Algebraic and Geometric Multiplicity of a characteristic root :-

Def :- Suppose 'A' is n×n matrix. If 'λ_1' is a characteristic root of order 't' of the characteristic equation of 'A', then 't' is called the algebraic multiplicity of 'λ_1'.

Def :- If 's' is the number of linearly independent characteristic vectors, corresponding to the repeated characteristic ~~vector~~ value 'λ_1', then 's' is called the "geometric multiplicity" of 'λ'.

Note: The geometric multiplicity of a characteristic root cannot exceed its Algebraic multiplicity. i.e., $\boxed{s \le t}$

Problems: ⊛ Find the eigen values and eigen vectors of the matrix 'A' and its inverse. where $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Sol^n:- Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of 'A' is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] = 0$$

$$\Rightarrow 1-\lambda = 0, \; 2-\lambda = 0, \; 3-\lambda = 0$$

$$\boxed{\lambda = 1} \quad \boxed{\lambda = 2} \quad \boxed{\lambda = 3}$$

$\therefore$ the characteristic roots are 1, 2, 3.

To find characteristic vector of '1' :- $(\lambda = 1)$

The eigen vector of 'A' is given by $(A - I)X = 0$

$$\Rightarrow \left\{ \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow 3x_2 + 4x_3 = 0 \;\Big|\; x_2 + 5x_3 = 0 \;\Big|\; 2x_3 = 0$$

Put $\boxed{x_1 = \alpha}$ $\quad \boxed{x_2 = 0}$ $\quad \boxed{x_3 = 0}$

It is arbitrary.

$$\therefore X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ where $\alpha \neq 0$ is the eigen vector corresponding to $\lambda = 1$.

To find characteristic vector of '2' $(\lambda = 2)$:-

The eigen vector of 'A' is given by $(A - 2I)X = 0$

$$\Rightarrow \left\{ \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow -x_1+3x_2+4x_3=0$, $5x_3=0$, $\boxed{x_3=0}$

$\rightarrow$ $\boxed{x_1=\cancel{0}}$ $3x_2+4x_3$ $\boxed{x_3=0}$

$\boxed{x_1=3x_2}$

put $\boxed{x_2=k}$ then $\boxed{x_1=3k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ the characteristic vector is $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

<u>To find characteristic vector of '3':-</u>

$(A-3I)X=0$ $\rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow -2x_1+3x_2+4x_3=0$, $-x_2+5x_3=0$

$\rightarrow 2x_1 = 15k+4k = 19k$ $\boxed{x_2=5x_3}$ put $\boxed{x_3=k}$ then $\boxed{x_2=5k}$

$\boxed{x_1=\frac{19}{2}k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{2}k \\ 5k \\ k \end{bmatrix} = k\begin{bmatrix} 19/2 \\ 5 \\ 1 \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$$

Hence eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ i.e., $1, 1/2, 1/3$. &

($\because$ the eigen values of A^{-1} are the reciprocals of the eigen values of A

eigen vectors of A^{-1} are same as eigen vectors of the matrix A.

* Determine the eigen values and eigen vectors of

$B = 2A^2 - \frac{1}{2}A + 3I$ where $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Soln:- we have $A^2 = A.A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 56 & -14 \\ 20 & -4 \end{bmatrix}$

$$\therefore B = 2A^2 - \frac{1}{2}A + 3I = \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}$$

characteristic eqⁿ of 'B' is $|B-\lambda I| = 0 \rightarrow \begin{vmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{vmatrix} = 0$

$\Rightarrow (111-\lambda)(-6-\lambda) + 3042 = 0$

$\rightarrow \lambda^2 + 105\lambda - 2376 = 0 \quad -666 - 111\lambda + 6\lambda + \lambda^2 + 3042 = 0$

$\Rightarrow (\lambda-33)(\lambda-72) = 0 \Rightarrow \boxed{\lambda = 33} \mid \boxed{\lambda = 72}$

$$\begin{array}{r} 39 \\ 78 \\ \hline 312 \\ 273\times \\ \hline 3042 \\ 666 \\ \hline 2376 \end{array}$$

∴ Eigen values of 'B' are 33 & 72.

For $\lambda = 33$:- The eigen vector of 'B' is given by $(B-33I)X = 0$

i.e, $\begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e, $x_1 = x_2$ (or) $\frac{x_1}{1} = \frac{x_2}{1}$

∴ the eigen vector for $\lambda = 33$, is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = 72$:- the eigen vector of 'B' is given by $(B-72I)X = 0$

i.e, $\begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 39x_1 - 78x_2 = 0$

$\boxed{x_1 = 2x_2} \rightarrow \frac{x_1}{2} = \frac{x_2}{1}$

∴ the eigen vector for $\lambda = 72$, is $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Hence the eigen vectors of B are $(1,1)^T$, $(2,1)^T$.

— * For the matrix 'A' $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3 + 5A^2 - 6A + 2I$.

Solⁿ:- the characteristic equation of 'A' is $|A - \lambda I| = 0$

i.e, $\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & -2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[(3-\lambda)(-2-\lambda)\right] - 2(0) - 3(0) = 0$

$\Rightarrow (1-\lambda)(-6-3\lambda+2\lambda+\lambda^2) = 0$

$\Rightarrow (1-\lambda)(\lambda^2-\lambda-6) = 0$

$\Rightarrow (1-\lambda)(\lambda^2-3\lambda+2\lambda-6) = 0$

$\Rightarrow (1-\lambda)\left[\lambda(\lambda-3)+2(\lambda-3)\right] = 0 \Rightarrow (1-\lambda)(\lambda-3)(\lambda+2) = 0$

$\Rightarrow \lambda = 1, -2, 3.$

W.K.T if 'λ' is an eigen value of 'A' and f(A) is a polynomial in 'A', then the eigen value of f(A) is f(λ).

Let $f(A) = 3A^3 + 5A^2 - 6A + 2I$

then, the eigen values of $f(A)$ are $f(1), f(3)$ & $f(-2)$

$\Rightarrow \therefore f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2(1) = 3+5-6+2 = 4$ [$\because$ eigen values of I a 1, 1, 1]

$\Rightarrow f(3) = 3(27) + 5(9) - 6(3) + 2 = 81+45-18+2 = 128-18 = 110$

$\Rightarrow f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2 = -24+20+12+2 = -24+34 = 10$

they, eigen values of $3A^3+5A^2-6A+2I$ are 4, 110, 10.

— * If 2, 3, 5 are the eigen values of 'A', then find the eigen values of $2A^3+3A^2+$ ...

— * Verify that the geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity given the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol^n:

the characteristic equation of 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (-2-\lambda)[(1-\lambda)(-\lambda) - 12] - 2(-2\lambda - 6) - 3(-4+1-\lambda) = 0$

$\Rightarrow (-2-\lambda)((\lambda-1)\lambda - 12) + 4\lambda + 12 - 3(-3-\lambda) = 0$

$\Rightarrow (-2-\lambda)(\lambda^2 - \lambda - 12) + 4\lambda + 12 + 9 + 3\lambda = 0$

$\Rightarrow 7\lambda + 21 - 2\lambda^2 + 2\lambda + 24 - \lambda^3 + \lambda^2 + 12\lambda = 0$

$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0 \Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\begin{array}{r|rrrr} \lambda = -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$-27+9+63-45$

$\Rightarrow (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0 \Rightarrow (\lambda+3)(\lambda^2 - 5\lambda + 3\lambda - 15) = 0$

$\Rightarrow (\lambda+3)[\lambda(\lambda-5) + 3(\lambda-5)]$

$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\Rightarrow \lambda = 5, -3, -3.$

$\therefore$ the characteristic roots are $5, -3, -3$.

Here '-3' is the multiple root of order '2'.

Hence the algebraic multiplicity of the characteristic root '-3' is '2'.

the characteristic roots corresponding to $\lambda=-3$ are

$\lambda=-3$:- $(A+3I)X=0 \Rightarrow \left(\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}+\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow$ ($x_1+2x_2-3x_3=0 \Rightarrow 2x_2=3x_3-x_1$

$2x_1+4x_2-6x_3=0 \Rightarrow 2x_2=3x_3-x_1$

$-x_1-2x_2+3x_3=0$ $\boxed{x_2=k}$

$\Rightarrow x_1=-2x_2+3x_3$

Put $x_3=k=x_2$

$x_1=-2k+3k$

$\boxed{x_1=k}$)

$x_1=-2k+3k$

$\Rightarrow \boxed{x_1=k}$

$\Rightarrow x_1+2x_2-3x_3=0$

$2x_1+4x_2-6x_3=0 \Rightarrow x_1+2x_2-3x_3=0$

$-x_1-2x_2+3x_3=0$

$\Rightarrow x_1=-2x_2+3x_3$

Put $x_2=\alpha,\ x_3=\beta$

$\Rightarrow x_1=-2\alpha+3\beta$

$\therefore \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-2\alpha+3\beta\\ \alpha\\ \beta\end{bmatrix}=\alpha\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+\beta\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$

thus the geometric multiplicity of $\lambda=-3$ is '2'.

thus here, geometric multiplicity = Algebraic multiplicity.

D: 12/7/13

Diagonalization of a matrix :-

A matrix 'A' is diagonalizable if $\exists$ an invertible matrix 'P' $\ni$ $P^{-1}AP=D$, where 'D' is diagonal matrix, where 'P' is said to be diagonalize 'A'.

Similarity of Matrix :- Let 'A' and 'B' be square matrices of order '[illegible]' then 'B' is said to be similar to 'A' if $\exists$ a non-singular matrix 'P' $\ni$ $\boxed{B=P^{-1}AP}$

thm :- An $n \times n$ matrix is diagonalizable $\Leftrightarrow$ if it possesses 'n' linearly independent eigen vectors.

Proof :- Let 'A' is diagonalizable, then 'A' is similar to a diagonal matrix $D = dia[\lambda_1, \lambda_2, \ldots, \lambda_n]$.

$\therefore \exists$ an invertible matrix 'P' $\ni$ $P^{-1}AP = D$

$$\Rightarrow AP = PD$$

$$\Rightarrow A[x_1, x_2, x_3, \ldots, x_n] = [x_1, x_2, \ldots, x_n] \cdot$$ 1H 33

dia$(\lambda_1, \lambda_2, \ldots$

$$\Rightarrow [Ax_1, Ax_2, \ldots Ax_n] = [\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots, \lambda_4 x_4]$$

$$\Rightarrow Ax_1 = \lambda_1 x_1, \; Ax_2 = \lambda_2 x_2, \ldots, Ax_n = \lambda_n x_n.$$

So, $x_1, x_2, \ldots, x_n$ are eigen vectors of 'A' corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively.

Since the matrix 'P' is non-singular its column vectors $x_1, x_2, \ldots, x_n$ are linearly independent.

$\therefore$ 'A' possesses 'n' linearly independent eigen vectors.

Conversely given that $x_1, x_2, \ldots, x_n$ be eigen vectors of 'A' corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively and these eigen vectors are L.I.

Define $P = (x_1, x_2, \ldots, x_n)$

Since the n-columns of 'P' are L.I, $|P| \neq 0$

Hence 'P^{-1}' exists.

Consider $AP = A[x_1, x_2, \ldots, x_n] = [Ax_1 \;\; Ax_2 \;\; \ldots \;\; Ax_n]$

$$= [\lambda_1 x_1 \;\; \lambda_2 x_2 \;\; \ldots \;\; \lambda_n x_n]$$

$$= [x_1 \;\; x_2 \;\; \ldots \;\; x_n] \begin{bmatrix} \lambda_1 & 0 \ldots & 0 \\ 0 & \lambda_2 \ldots & 0 \\ 0 & \ldots & \lambda_n \end{bmatrix}$$

$$= PD$$

$\therefore \boxed{AP = PD}$ where $D = diag(d_1, d_2, \ldots, d_n)$

$\Rightarrow P^{-1}(AP) = P^{-1}(PD)$

$\Rightarrow P^{-1}(AP) = (P^{-1}P)D$

$\Rightarrow P^{-1}AP = D = diag(d_1, d_2, \ldots, d_n)$

Hence Proved.

$\boxed{\text{Note}}$: Suppose 'A' is real symmetric matrix with 'n' pairwise distinct eigen values $d_1, d_2, \ldots, d_n$. then the corresponding eigen vectors $X_1, X_2, \ldots, X_n$ are pair-wise orthogonal. $P = (e_1, e_2, \ldots e_n)$

$e_1 = \frac{X_1}{\|X_1\|}, e_2 = \frac{X_2}{\|X_2\|}, \ldots e_n = \frac{X_n}{\|X_n\|}$

then 'P' will be orthogonal matrix i.e., $P^{-1}P = PP^T = I \Rightarrow \boxed{P^{-1} = P^T}$

$\therefore \boxed{P^{-1}AP = D} \Rightarrow \boxed{P^TAP = D}$

MODAL and Spectral matrices :-

Def :- The matrix 'P', which diagonalise the square matrix 'A' is called the "modal-matrix" of 'A' and the resulting diagonal matrix 'D' is known as "Spectral matrix".

Note: (1). The diagonal elements of 'D' are the eigen values of 'A' & they occur in the same order as is the order of their corresponding eigen vectors in the column vectors of 'P'.

2). If the eigen values of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

CALCULATION OF POWERS OF A MATRIX :-

By using the diagonalisation, we can obtain the Powers of a matrix.

Procedure

- * Determine eigenvalues of 'A'.
- * Find eigenvectors and write the modal matrix 'P'.
- * Find the diagonal matrix 'D' from $D = P^{-1}AP$.
- * compute A^n from $A^n = PD^nP^{-1}$

Let 'A' be the square matrix then a non-singular matrix 'P'

$\Rightarrow \boxed{D = P^{-1}AP}$

$D^2 = D.D = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P \rightarrow \boxed{D^2 = P^{-1}A^2P}$ $(\because PP^{-1} = I)$

$\|^{ly}$ $\boxed{D^3 = P^{-1}A^3P}$ ---- In general $\boxed{D^n = P^{-1}A^nP}$ ——— (1)

then $PP^T = P^TP = I \Rightarrow \boxed{P^T = P^{-1}}$ ($\because$ 'A' is Symmetric)

$\therefore$ Diagonalised matrix $= P^{-1}AP = P^TAP$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \\ 0 & -3/\sqrt{3} & 12/\sqrt{6} \\ -2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$\boxed{\text{Note}}$: If 'A' is non-singular matrix, and its eigen values are distinct then the matrix 'P' is found by grouping the eigen vectors of 'A' into square matrix and the diagonal matrix has the eigen values of 'A' as its elements.

* If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find (a). A^8 (b). A^4

Sol^n:- The characteristic eq^n of 'A' is $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-4\right] - 1\left[4\right] + 4(2-\lambda) = 0$

$\Rightarrow (1-\lambda)(6-2\lambda-3\lambda+\lambda^2-4) - 4 + 8 - 4\lambda = 0$

$\Rightarrow (\lambda^2-5\lambda+2)(1-\lambda) - 4\lambda + 4 = 0 \Rightarrow \lambda^2-5\lambda+2-\lambda^3+5\lambda^2-2\lambda-4\lambda+4=0$

$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$

$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$

$(1-6+11-6 = 12-12=0)$

$$\begin{array}{c|cccc} \lambda=1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\Rightarrow (\lambda-1)(\lambda^2-5\lambda+6)=0$

$\Rightarrow (\lambda-1)(\lambda^2-3\lambda-2\lambda+6)=0 \Rightarrow (\lambda-1)\left[\lambda(\lambda-3)-2(\lambda-3)\right]=0$

$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)=0$

$\Rightarrow \lambda_1 = 1 \mid \lambda_2 = 2 \mid \lambda_3 = 3.$

Characteristic vector corresponding to $\lambda=1$:-

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} y+z=0 \Rightarrow \boxed{y=-z} \\ y+z=0 \\ -4x+4y+2z=0 \Rightarrow -4x-4k+2k=0 \\ \Rightarrow -4x-2k=0 \Rightarrow 4x=-2k \end{array}$$

Put $\boxed{\begin{array}{l} z=k \\ y=-k \end{array}}$

$\boxed{x=-\frac{k}{2}}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k/2 \\ -k \\ k \end{bmatrix} = -\frac{k}{2}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=1$.

Characteristic vector corresponding to $\lambda=2$:-

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} -x+y+z=0 \\ z=0 \\ -4x+4y+z=0 \\ \rightarrow -4x=-4y \rightarrow \boxed{x=y} \end{array}$$

Put $\boxed{y=k=x}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=2$.

Characteristic vector corresponding to $\lambda=3$:-

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -2x+y+z=0 \\ -y+z=0 \rightarrow \boxed{y=z} \\ -4x+4y=0 \Rightarrow \boxed{x=y} \end{array}$$

Put $z=k$, $y=x=k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=3$.

Consider $P=[X_1 \;\; X_2 \;\; X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

We have $|P|=-1$ and $P^{-1} = -\begin{bmatrix} 1 & -1 & 0 \\ -4 & 3 & 1 \\ 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

Now $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = diag(1,2,3) = D$ (say)

(a). $\therefore D^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 3^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix}$

$A^8 = PD^8P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 256 & 6561 \\ 2 & 256 & 6561 \\ -2 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

$\Leftarrow \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$

(b). $D^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$

$\therefore A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 16 & 81 \\ 2 & 16 & 81 \\ -2 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$

* Find a matrix 'P' which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate A^4. Find the eigen values and eigen vectors of 'A'.

Solⁿ:- characteristic equation of 'A' is given by $|A - \lambda I| = 0$

i.e, $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right]-0-1\left[2-2(2-\lambda)\right]=0$$

$$\Rightarrow (1-\lambda)\left(6-5\lambda+\lambda^2-2\right)-\left(2-4+2\lambda\right)=0$$

$$\Rightarrow \left(\lambda^2-5\lambda+4\right)(1-\lambda)-\left(2\lambda-2\right)=0 \Rightarrow \lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda-2\lambda+2=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0 \qquad \Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0 \quad \underline{\lambda=1}$$

$$1-6+11-6=12-12=0$$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & & -5 & 6 & 0 \end{array}$$

$$\rightarrow (\lambda-1)(\lambda^2-5\lambda+6)=0$$

$$\Rightarrow (\lambda-1)(\lambda^2-3\lambda-2\lambda+6)=0 \Rightarrow (\lambda-1)\{\lambda(\lambda-3)-2(\lambda-3)\}=0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)=0$$

thus the eigen values of 'A' are 1,2,3 $\boxed{\lambda=1,2,3.}$

$\underline{\lambda=1}$:-

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow -x_3=0 \rightarrow \boxed{x_3=0}$$

$$x_1+x_2+x_3=0 \rightarrow \boxed{x_1=-x_2}$$

$$2x_1+2x_2+2x_3=0 \quad \text{Put } \boxed{x_2=k}, \ \boxed{x_1=-k}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -k \\ k \\ 0 \end{bmatrix}=k\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda=2}$:-

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1-x_3=0 \Rightarrow \boxed{x_3=-x_1} \Rightarrow \boxed{x_3=2k}$$

$$x_1+x_3=0$$

$$2x_1+2x_2+x_3=0 \Rightarrow 2x_2+x_1=0$$

$$2x_2=-x_1$$

$$\boxed{x_1=-2x_2}$$

$$\text{Put } \boxed{x_2=k} \Rightarrow \boxed{x_1=-2k}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -2k \\ k \\ 2k \end{bmatrix}=k\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$\underline{\lambda=3}$:-

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow -2x_1-x_3=0 \Rightarrow \boxed{x_3=2k}$$

$$x_1-x_2+x_3=0$$

$$2x_1+2x_2=0$$

$$\boxed{x_1=-x_2}$$

$$\boxed{x_2=k}, \ \boxed{x_1=-k}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -k \\ k \\ 2k \end{bmatrix}=k\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore P=\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}=\text{modal matrix of 'A'.}$$

$$|P| = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 1(2-2)+2(-2)-1(-2) = -4+2 = -2 \neq 0$$

$\therefore$ 'P^{-1}' exists.

$$\therefore P^{-1} = \frac{1}{-2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

Now $P^{-1}AP = \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 1/2 \\ -2 & -2 & 0 \\ 3 & 3 & 3/2 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Hence $A^4 = PD^4P^{-1}$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 32 & -81 \\ -1 & 16 & 9 \\ 0 & 32 & 162 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

— * Diagonalize the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Solⁿ:- the characteristic eqⁿ of 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (8-\lambda)\left[(-3-\lambda)(1-\lambda)+8\right] + 8\left[4-4\lambda+6\right] - 2(-16+9+3\lambda) = 0$$

$\Rightarrow (8-\lambda)(-3+3\lambda-\lambda+\lambda^2-8) + 8(4\lambda+10) - 2(3\lambda-7)) =$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

3	1	-6	11	-6
	0	3	-9	6
	1	-3	2	0

$\lambda^2 - 3\lambda + 2$

$\lambda^2 - 2\lambda - \lambda + 2$

$\lambda(\lambda-2) - (\lambda-2)$

$\Rightarrow \lambda = 3, 1, 2$

$$\therefore D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

04N1, 05P2 05N9
04P5, 04N5 04P4.
1258, 05P5
05P0, 04P0

Nilpotent matrix :- A non-zero matrix 'A' is said to be nilpotent if for some positive n', $\boxed{A^n = O}$.

Note ①. A non-zero matrix is nilpotent $\Leftrightarrow$ all its eigen values are equal to zero.

②. A non-zero nilpotent matrix cannot be similar to a diagonal matrix (i.e.,) it cannot be diagonalised.

eg:- P.T the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

sol:- Given $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus 'A' is nilpotent & hence cannot diagonalised.

(or)

The characteristic eqn of 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \Rightarrow \left| \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \right| = 0 \Rightarrow \lambda^2 = 0$$

$\Rightarrow \lambda = 0, 0$. are the characteristic values.

for $\lambda = 0$:- the characteristic vector is given by $(A - \lambda I)X = 0$
$\Rightarrow AX = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0$$

$x_1 = k$ (say)

$\therefore$ the characteristic vector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

the given matrix has only one linearly independent characteristic vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to repeated characteristic value $(\lambda = 0)$ '0'.

$\therefore$ the matrix is not diagonalizable.

— * S.T the matrix 'A' is cannot be diagonalized.

where $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Sol:- Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic Eqn of 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

$\lambda = 2, 2, 1$

$\therefore$ 2, 2, 1 are the characteristic values of 'A'.

Since the eigen values of 'A' are not distinct,
$\therefore$ Eigen vectors of 'A' are not linear independent.
Hence the matrix 'A' is not diagonalised.

<u>characteristic Vectors</u> :-

The characteristic vector corresponding to $\lambda = 2$ is given by $(A - \lambda I)X = 0$

$(A - 2I)X = 0$. $(\because \lambda = 2)$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 = R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_2 + 4x_3 = 0$; $-x_3 = 0$; Put $\boxed{x_1 = k}$

$\Rightarrow 3x_2 = -4x_3$ $\quad\Rightarrow \boxed{x_3 = 0}$

$\Rightarrow 3x_2 = -4k(0)$

$\boxed{x_2 = 0}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$$

The Eigen vector corresponding to the eigen value $\lambda = 1$ is given by $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(\because \lambda = 1)$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ by solving $x_1 + 3x_2 + 4x_3 = 0$; $x_2 - x_3 = 0$

$\boxed{x_1 = -7x_3}$ $\quad \boxed{x_2 = x_3}$

$\boxed{x_1 = -7k}$ $\quad$ Put $\boxed{x_3 = k}$

$\boxed{x_2 = k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}.$$

$\therefore$ Modal matrix $(P) = [x_1 \ x_2 \ x_3] = \begin{bmatrix} 1 & 1 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad\therefore$ 'A' cannot be diagonalised

then $\boxed{|P| = 0}$ $\therefore$ 'P^{-1}' does not exist $\therefore D = P^{-1}AP$ also does not exist.

—* CAYLEY-HAMILTON THEORM *—

Matrix Polynomial :— An expression of the form $F(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m$; $A_m \neq 0$, where $A_0, A_1, A_2, \ldots, A_m$ are matrices each of order $n \times n$ over a field 'F', is called a "matrix Polynomial" of degree 'm'.

Equality of matrix Polynomials :— Two matrix Polynomials are equal $\Leftrightarrow$ the coefficients of like powers of 'x' are the same.

Thm :— Every square matrix satisfies its own characteristic eqn.

Proof :— Let 'A' be n-rowed square matrix. then

$|A - \lambda I| = 0$ is the characteristic equation of 'A'.

Let $|A - \lambda I| = (-1)^n \left[\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + a_3 \lambda^{n-3} + \cdots + a_n\right]$

Since all the elements of $A - \lambda I$ are almost of first degree in 'λ', all the elements of $\operatorname{adj}(A - \lambda I)$ are polynomials in 'λ' of degree $(n-1)$ or less and hence $\operatorname{adj}(A - \lambda I)$ can be written as a matrix-polynomials in 'λ'.

Let $\operatorname{adj}(A - \lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-2} \lambda^1 + B_{n-1}$

Where $B_0, B_1, \ldots, B_{n-1}$ are n-rowed matrices.

Now $(A - \lambda I) \operatorname{adj}(A - \lambda I) = (A - \lambda I) \operatorname{adj}(A - \lambda I)$.

$= |A - \lambda I| \, I_n$

$(A - \lambda I)\left(B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-2} \lambda^1 + B_{n-1}\right) = (-1)^n \left(\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n\right) I_n$

Comparing coefficients of like powers of λ, we obtain

$-B_0 = (-1)^n I$

$AB_0 - B_1 = (-1)^n a_1 I$

$AB_1 - B_2 = (-1)^n a_2 I$ - - - - - -

$AB_{n-1} = (-1)^n a_n I$

(45)

$\lambda = 4$:- $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = 0$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$\boxed{x_2 = 0}$ $\boxed{x_3 = k,\ x_1 = -k}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 4+\sqrt{2}$:- $\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -\sqrt{2}x_1 + x_2 = 0$, $x_2 - \sqrt{2}x_3 = 0$, $x_1 - \sqrt{2}x_2 + x_3 = 0$

$\cancel{x_2} - \sqrt{2}x_1 = 0$

$\cancel{x_2} - \sqrt{2}x_3 = 0$

$-\quad +$

$\sqrt{2}x_3 - \sqrt{2}x_1 = 0$

$x_3 - x_1 = 0$

$\boxed{x_3 = x_1} = k$

$2x_1 - \sqrt{2}x_2 = 0$

$2x_1 = \sqrt{2}x_2$

$\boxed{x_2 = \sqrt{2}k}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ \sqrt{2}k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

$\lambda = 4-\sqrt{2}$:- $\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \sqrt{2}x_1 + x_2 = 0$, $x_1 + \sqrt{2}x_2 + x_3 = 0$, $\cancel{x_2} + \sqrt{2}x_3 = 0$

$\rightarrow 2k = -\sqrt{2}x_2$

$\boxed{x_2 = -\sqrt{2}k}$

$\cancel{x_2} + \sqrt{2}x_1 = 0$

$\boxed{x_3 = x_1} = k$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -\sqrt{2}k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$

— * $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Soln:- the characteristic eqn of 'A' is $|A - \lambda I| = 0$

$\rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\rightarrow (1-\lambda)[(3-\lambda)^2 - 1] = 0$

$\Rightarrow (1-\lambda)(\lambda^2 + 9 - 6\lambda - 1) = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$

$\rightarrow (1-\lambda)(\lambda^2 - 4\lambda - 2\lambda + 8) = 0$

$\rightarrow \lambda = 1,\ \lambda(\lambda-4) - 2(\lambda-4) = 0$

By Cayley-Hamilton thm, we must have

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{——— (1)}$$

$$A^2 = A \cdot A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

To find A^{-1}: $(1) \Rightarrow A^{-1}\left[A^3 - 5A^2 + 7A - 3I\right] = 0$

$$\Rightarrow A^2 - 5A + 7I = 3A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\left(A^2 - 5A + 7I\right)$$

$$\Rightarrow A^{-1} = \frac{1}{3}\left\{\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right\} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

$$(1) \Rightarrow A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$\Rightarrow A^4 = 5A^3 - 7A^2 + 3A$$

$$= 5\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 7\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 3\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -49 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

- * Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (33)

Sol^n:- The characteristic eq^n of 'A' is $|A - \lambda I| = 0$

i.e, $\begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0 \rightarrow (8-\lambda)\left[(-3-\lambda)(1-\lambda) - 8\right] + 8(4-4\lambda+6) + 2(-16+9+3\lambda) = 0$

$\rightarrow (8-\lambda)(-3+3\lambda-\lambda+\lambda^2-8) + 8(-4\lambda+10) + 2(3\lambda-7) = 0$

$\rightarrow (8-\lambda)(\lambda^2+2\lambda-11) + 8(10-4\lambda) + 2(3\lambda-7) = 0$

$\rightarrow 8\lambda^2+16\lambda-88-\lambda^3-2\lambda^2+11\lambda+80-32\lambda+6\lambda-14 = 0$

$\rightarrow -\lambda^3+6\lambda^2+\lambda-22 = 0 \Rightarrow \lambda^3-6\lambda^2-\lambda+22 = 0$

To verify the Cayley-Hamilton thm, we have to prove that

$\Rightarrow A^3 - 6A^2 - A + 22I = 0$

Now $A^2 = A \cdot A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$

$\therefore A^3 - 6A^2 - A + 22I = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} - 6\begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + 22\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore A^3 - 6A^2 - A + 22I = 0$

$\therefore$ Hence verified.

2/3/15

ECE-C(o):- (A), C9, D2, D8, D9, E4, E5, ~~F4~~, G0, G7, H0, H5, 37

(4) Verify cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and hence find 'A^{-1}' and $B = A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$?

Sol:- Given $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

the characteristic Eqn of 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0.$$

Verification :-

By cayley-Hamilton thm, Every square matrix satisfies its own characteristic Eqn.

i.e, $A^2 - 4A - 5 = 0$ ——— (1)

Now $A^2 = A.A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

(1) $\Rightarrow A^2 - 4A - 5I = 0$

$$\Rightarrow \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow 0 = 0.$$

$\therefore$ cayley-Hamilton thm is Verified.

Find A^{-1}:- multiply 'A^{-1}' on b.s of Eqn (1), we get

$\Rightarrow A^{-1}(A^2 - 4A - 5) = 0A^{-1}$

$\Rightarrow A - 4I - 5A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{5}[A - 4I] = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

Find "B" :— Given $B = A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= (A^5 - 4A^4 - 5A^3 - 2A^3) + 11A^2 - A - 10I$

$= A^3(A^2 - 4A - 5I) - 2A^3 + 11A^2 - A - 10I$

$= A^3(0) - 2A^3 + 11A^2 - A - 10I$ $[\because (1)]$

$B = -2A^3 + 11A^2 - A - 10I$ —————— (2).

Here $A^2 = A \cdot A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix}$

$(2) \Rightarrow B = -2 \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix} + 11 \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$ //.

HW.

5. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ find the value of the matrix

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

6. Verify cayley-Hamilton theorem and find inverse of

$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Orthogonal reduction to real-Symmetric matrices [in Diagonalisation]:

Suppose 'A' is real Symmetric matrix with 'n' pairwise distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. then the corresponding eigen vectors $X_1, X_2, \ldots, X_n$ are pairwise orthogonal. $P = (e_1, e_2, \ldots, e_n)$

$e_1 = \frac{X_1}{\|X_1\|}$, $e_2 = \frac{X_2}{\|X_2\|}$, ---, $e_n = \frac{X_n}{\|X_n\|}$ then 'P' will be orthogonal-matrix. i.e, $PP^T = P^TP = I \Rightarrow P^{-1} = P^T$.

$\therefore \boxed{P^{-1}AP = D} \Rightarrow \boxed{P^TAP = D}$

$\therefore$ Diagonalised matrix $P^{-1}AP = P^TAP$.

①. Diagonalize the matrix, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, by orthogonal reduction.

Ans:- The characteristic Eq^n is $|A - \lambda I| = 0$.

i.e, $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\left[(3-\lambda)^2 - 1\right] = 0 \quad \Rightarrow \lambda = 1, 2, 4.$

we can find the eigen vectors corresponding to the eigen values as

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Model matrix $(P) = \begin{bmatrix} \frac{x_1}{\|X_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1^2+0^2+0^2}} & 0/\sqrt{0^2+1^2+1^2} & 0/\sqrt{0^2+1^2+1^2} \\ \frac{0}{\sqrt{1^2+0^2+0^2}} & 1/\sqrt{0^2+1^2+1^2} & -1/\sqrt{0^2+1^2+1^2} \\ 0/\sqrt{1^2+0^2+0^2} & 1/\sqrt{0^2+1^2+1^2} & 1/\sqrt{0^2+1^2+1^2} \end{bmatrix}$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Then $P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Now check $PP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore \boxed{PP^T = I}$ $\quad$ ||ly we get $\boxed{P^TP = I}$

$\therefore \boxed{P^T = P^{-1}}$

$$\therefore P^T = P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

we can easily verify that $PAP^{-1} = D$ (or) $D = P^{-1}AP$

$$\therefore D = \text{diag}(1, 2, 4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

thus 'A' is reduced to diagonal form by orthogonal reduction.

(2). Find the Diagonal matrix orthogonally similar to the following real symmetric matrix. Also obtain the transforming matrix.

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}.$$

Sol:- The characteristic Eqn of 'A' is

$$\begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)\left[(-8-\lambda)(-8-\lambda)-1\right]-4\left[4(-8-\lambda)-4\right]-4\left[-4-4(8+\lambda)\right]=0$$

$$\Rightarrow (7-\lambda)\left[(8+\lambda)^2-1\right]-4\left[-32-4\lambda-4\right]-4\left[-4-32-4\lambda\right]=0$$

$$\Rightarrow (7-\lambda)\left[64+\lambda^2+16\lambda-1\right]-4\left[-4\lambda-36\right]-4\left[-4\lambda-36\right]=0$$

$$\Rightarrow (7-\lambda)\left[16\lambda+\lambda^2+63\right]+(4\lambda+36)\left[4+4\right]=0$$

$$\Rightarrow (7-\lambda)(\lambda^2+16\lambda+63)+(4\lambda+36)(8)=0$$

$$\Rightarrow 7\lambda^2+112\lambda+441-\lambda^3-16\lambda^2+63\lambda+32\lambda+288=0$$

$$\Rightarrow -\lambda^3-9\lambda^2+81\lambda+729=0$$

$$\Rightarrow \lambda^3+9\lambda^2-81\lambda-729=0. \quad \left\{\because \text{put } \boxed{\lambda=9} \Rightarrow \cancel{729}-9(\cancel{81})-\cancel{81}(9)-\cancel{729}=0 \Rightarrow 0=0\right\}$$

by Synthetic Division (or) Horner's method

$\lambda=9$	1	9	-81	-729
	0	9	162	729
	1	18	81	0

$\Rightarrow (\lambda-9)(\lambda^2+18\lambda+81)=0.$

$\Rightarrow (\lambda-9)(\lambda^2+9\lambda+9\lambda+81)=0$

$\Rightarrow (\lambda-9)\left[\lambda(\lambda+9)+9(\lambda+9)\right]=0$

$\Rightarrow (\lambda-9)(\lambda+9)(\lambda+9)=0$

$\therefore \lambda=9,-9,-9.$ are the eigen values.

Eigen vector corresponding to $\lambda=9$:-

the characteristic vector corresponding to $\lambda=9$ is given by

$(A-9I)\lambda=0 \quad [\because (A-\lambda I)x=0]$

$$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2=R_2+2R_1 \quad ; \quad R_3=R_3-2R_1$

$$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & -9 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3=R_3-R_2$

$$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1+4x_2-4x_3=0 \quad ; \quad -9x_2-9x_3=0$

$-\cancel{9}x_2=\cancel{9}x_3 \quad$ put $\boxed{x_3=k}$

$-2x_1=4k+4k$

$\boxed{x_2=-x_3} \Rightarrow \boxed{x_2=-k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = -9$:-

$$(A - \lambda I)X = 0$$

$$\Rightarrow (A + 9I)X = 0$$

$$\Rightarrow \begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = 4R_2 - R_1$; $R_3 = 4R_3 + R_1$

$$\Rightarrow \begin{bmatrix} 16 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$16x_1 + 4x_2 - 4x_3 = 0$; put $\boxed{x_3 = k_1}$ & $\boxed{x_2 = k_2}$

$$\Rightarrow 16x_1 = 4x_3 - 4x_2$$

$$16x_1 = 4k_1 - 4k_2$$

$$\boxed{x_1 = \frac{k_1 - k_2}{4}}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_1 - k_2}{4} \\ k_2 \\ k_1 \end{bmatrix} = \frac{k_1}{4} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + \frac{k_2}{4} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ are two vectors corresponding $\lambda = -9$.

above two vectors are not orthogonal.

so, we can write the linear combination

Consider $a\begin{bmatrix}1\\0\\4\end{bmatrix}+b\begin{bmatrix}-1\\4\\0\end{bmatrix}=\begin{bmatrix}1\\0\\4\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a-b\\4b\\4a\end{bmatrix}=\begin{bmatrix}1\\0\\4\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a-b\\4b\\4a\end{bmatrix}$ is orthogonal to $\begin{bmatrix}1\\0\\4\end{bmatrix}$ —— (1)

$\Rightarrow (a-b)+4b\times 0+4a\times 4=0$

$\Rightarrow a-b+16a=0$

$\Rightarrow \boxed{b=17a}$

from (1) $\Rightarrow$ $\begin{bmatrix}-16a\\68a\\4a\end{bmatrix}$ is orthogonal to $\begin{bmatrix}1\\0\\4\end{bmatrix}$

$\therefore 4a\begin{bmatrix}-4\\17\\1\end{bmatrix}$ is orthogonal to $\begin{bmatrix}1\\0\\4\end{bmatrix}$

$\therefore \begin{bmatrix}-4\\17\\1\end{bmatrix}\begin{bmatrix}1\\0\\4\end{bmatrix}$ & $\begin{bmatrix}-4\\-1\\1\end{bmatrix}$ are pairwise orthogonal

vectors. Now Normalizing above vectors, we get

$$P = \begin{bmatrix} -4/\sqrt{306} & 1/\sqrt{17} & -4/\sqrt{18} \\ 17/\sqrt{306} & 0 & -1/\sqrt{18} \\ 1/\sqrt{306} & 4/\sqrt{17} & 1/\sqrt{18} \end{bmatrix}$$ is the required orthogonal matrix that will diagonalise 'A'.

then $PP^T = P^TP = I \Rightarrow \boxed{P^T = P^{-1}}$ ($\because$ 'A' is Symmetric)

$\therefore$ Diagonalised matrix $= P^{-1}AP = P^TAP$

$$D = \begin{bmatrix} \frac{-4}{\sqrt{306}} & 17/\sqrt{306} & 1/\sqrt{306} \\ 1/\sqrt{17} & 0 & 4/\sqrt{17} \\ -4/\sqrt{18} & -1/\sqrt{18} & 1/\sqrt{18} \end{bmatrix} \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix} \begin{bmatrix} -4/\sqrt{306} & 1/\sqrt{17} & \frac{-4}{\sqrt{18}} \\ 17/\sqrt{306} & 0 & -1/\sqrt{18} \\ 1/\sqrt{306} & 4/\sqrt{17} & 1/\sqrt{18} \end{bmatrix}$$

$$D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix} = \text{diag}(9, -9, -9)$$

③. Determine the modal matrix 'P' for $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ & Hence diagonalize 'A' by orthogonal reduction.

④. Find an orthogonal matrix that will diagonalize the real symmetric matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ also find resulting diagonal matrix.

eigen vectors :-

$\underline{\lambda = -3}$:- $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_1 + (3+4i)x_2 = 0 \Rightarrow x_1 = \left(\frac{-3+4i}{5}\right)x_2 \Rightarrow \frac{x_1}{-3-4i} = \frac{x_2}{5}$

$(3-4i)x_1 + 5x_2 = 0 \Rightarrow \frac{x_1}{-5} = \frac{x_2}{3-4i}$

$\therefore$ eigen vector is $\begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$ //

$\underline{\lambda = 7}$:- $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -5x_1 + (3+4i)x_2 = 0$

$\Rightarrow 5x_1 = (3+4i)x_2$

$\Rightarrow \frac{x_1}{3+4i} = \frac{x_2}{5}$

$\therefore$ eigen vector is $\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$ //

Q. Find the eigen value of the matrix $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Soln:- The characteristic matrix of 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3i-\lambda & 2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0 \Rightarrow (3i-\lambda)(-i-\lambda) - (2+i)(-2+i) = 0$$

$\Rightarrow (3 - 3i\lambda + i\lambda + \lambda^2) - (-4 + \cancel{2i} - \cancel{2i} - 1) = 0$

$\Rightarrow 8 - 2i\lambda + \lambda^2 = 0$

$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0 \Rightarrow \lambda^2 - 4i\lambda + 2i\lambda + 8 = 0$

$\Rightarrow \lambda(\lambda - 4i) + 2i(\lambda - 4i) = 0 \Rightarrow \boxed{\lambda = 4i, -2i}$ //

(48)

3). S.T $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is a skew-Hermitian matrix and also unitary. Find the Eigen values corresponding to the eigen vectors of 'A'.

Sol^n:- $A^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

$\therefore$ $\boxed{A^T = -\bar{A}}$ $\Rightarrow$ 'A' is a skew-Hermitian matrix.

Now we P.T 'A' is unitary. We have to S.T $A(\bar{A})^T = (\bar{A})^T A = I$

$(\bar{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

$\therefore$ $A(\bar{A})^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$(\bar{A})^T A = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ $A(\bar{A})^T = (\bar{A})^T A = I$.

Hence 'A' is unitary matrix.

The characteristic eq^n of 'A' is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow (i-\lambda)(\lambda^2+1) = 0$

$\Rightarrow \boxed{\lambda = -i}$; $\lambda^2 = -1$

$\boxed{\lambda = \pm i}$

eigen Vector :-

$\lambda = i$:- $(A - \lambda I)x = 0$

$$\begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -ix_2 + ix_3 = 0$ & $ix_2 + (-ix_3) = 0$

$\Rightarrow \boxed{x_2 = x_3}$

put $\boxed{x_1 = k}$ & $\boxed{x_2 = x_3 = k_1}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k_1 \\ k_1 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = -i$:- $(A - \lambda I)x = 0$

$$\begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \boxed{x_1 = 0}$

$x_2 + x_3 = 0$ put $\boxed{x_2 = k}$

$x_3 = -x_2 \Rightarrow \boxed{x_3 = -k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ -k \end{bmatrix} = k \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

④. P.T $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix. (49)

Sol:- Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \Rightarrow \bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$

$(\bar{A})^T = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1+i \end{bmatrix}$

$\Rightarrow A(\bar{A}^T) = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1+i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow (\bar{A})^T A = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore (\bar{A})^T A = A(\bar{A})^T = I$

$\therefore$ 'A' is unitary matrix //

⑤. S.T $A = \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$ is unitary if $a^2+b^2+c^2+d^2 = 1$.

Sol:-

$\bar{A} = \begin{pmatrix} a-ic & -b-id \\ b-id & a+ic \end{pmatrix}$

$(\bar{A})^T = \begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

$\therefore$ Given $A(\bar{A})^T = I$

$\Leftrightarrow \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}\begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix} = I$

$\Leftrightarrow \begin{bmatrix} a^2+c^2+b^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{bmatrix} = I$

If $\boxed{a^2+b^2+c^2+d^2 = 1}$ //

Quardratic forms :-

An expression of the form $\Phi = X^T A X =$
$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \quad \text{—— (1)}$$
where a_{ij} are constant is called a "Quardratic-form" in "n-variables".

The constant a_{ij} are real numbers then the quardratic form is called "real quardratic form".

Here 'A' is called "Symmetric matrix" (or) "Coefficient matrix".

The standard form of Q.F of three variables x, y, z is
$$ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fzy = 0 \quad \text{—— (2)}$$
(or) for three variables x_1, x_2, x_3 is
$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 = 0 \quad \text{—— (2)}$$

$\boxed{\text{Note}}$: From the above eq^n(2) the symmetric matrix 'A' (or) the co-efficient matrix 'A'.

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \quad \text{(or)} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here $a_{12} = a_{21}$
$a_{13} = a_{31}$
$a_{23} = a_{32}$

$\boxed{\text{Note}}$: The standard form of Q.F of four variable x, y, z, t is
$$A = \begin{bmatrix} a & h & g & f \\ h & b & p & q \\ g & p & c & r \\ f & q & r & d \end{bmatrix}$$
$$ax^2 + by^2 + cz^2 + dt^2 + 2hxy + 2gxz + 2fxt + 2pyz + 2qyt + 2rzt = 0.$$

— * The Q.F of the Symmetric matrix

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$ is

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $X^T = \begin{bmatrix} x & y & z \end{bmatrix}$

The Q.F is $= X^T A X$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{bmatrix}$$

$$= ax^2 + hyx + zgx + hxy + by^2 + fzy + zgx + zfy + cz^2$$

$$= ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz = 0$$

<u>Problems</u> :- ①. Find the Symmetric matrix corresponding to the Q.F $x^2 + y^2 + 3z^2 + 4xy + 5yz + 6zx = 0$. —— (1)

<u>Sol</u>:- W.k.t Q.F eqn for 3-variables is

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ —— (2)

Comparing (1) & (2)

$a=1$, $b=1$, $c=3$,

$2h=4$; $2f=5$; $2g=6$

$h=2$; $f=5/2$; $g=3$.

∴ Symmetric matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$$

(15)

② Find the Symmetric matrix Corresponding to the Q.F $x_1^2 + 2x_2^2 + 4x_2x_3 + x_3x_1 = 0$ —(1)

Sol:- The standard form of 3-variables x_1, x_2, x_3 is

$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 = 0$ —(2)

Comparing (1) & (2)

$a_{11} = 1$; $2a_{12} = 0 \Rightarrow a_{12} = 0$

$a_{22} = 2$; $2a_{23} = 4 \Rightarrow a_{23} = 2$

$a_{33} = 0$; $2a_{31} = 1 \Rightarrow a_{31} = 1/2$

$$\therefore A = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 2 \\ 1/2 & 2 & 0 \end{bmatrix}$$

③ Find Q.F Corresponding to the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

Sol:- Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $X^T = \begin{bmatrix} x & y & z \end{bmatrix}$

The required Quadratic form $Q = X^TAX$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x+2y+3z \\ 2x+y+3z \\ 3x+3y+z \end{bmatrix}$$

$$= x^2 + 2xy + 3zx + 2xy + y^2 + 3yz + 3xz + 3yz + z^2$$

$$= x^2 + y^2 + z^2 + 4xy + 6xz + 6yz = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$\boxed{\text{Note}}$: The standard form of Q.F of four variables x_1, x_2, x_3, x_4 is $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{14}x_1x_4 + 2a_{23}x_2x_3 + 2a_{24}x_2x_4 + a_{34}x_3x_4 = 0$

(4) Index of the Q.F :-

* In the Q.F the no. of +ve terms is called "index of the Q.F." It is denoted by 's'.

* Signature of the Q.F :- In the Q.F the excess no. of +ve terms over the no. of -ve terms is called "signature of the Q.F." i.e, $\boxed{2s-r}$

$\therefore \boxed{\text{signature} = 2s - r}$

where s is index
r - rank.

* nature of the Q.F :- The Q.F $X^T A X$ in 'n'-variables is said to be

(i). +ve definite :- If $r = n$ & $s = n$ (or) If the all eigen values of 'A' are +ve (or) In Q.F the no. of terms all are +ve.

(ii). -ve definite :- If $r = n$ & $s = 0$ (or) If the all eigen values of 'A' are -ve (or) In Q.F the no. of terms all are -ve.

(iii). +ve semi definite :- If $r < n$ & $s = r$ (or) If the all the eigen value of $A \geq 0$, atleast one eigen value is zero (or) In the Q.F atleast one term are missing remaining all terms are +ve.

(iv). -ve semi definite :- If $r < n$ & $s = 0$ (or) If all the eigen value of $A \leq 0$, at least one eigen value is zero (or) In the Q.F atleast one term are missing remaining all terms

(56)

duce the Q.F $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$ into form of square (or) Normal form by using orthogonal transformation (or) orthogonal reduction and give the matrix of transformation. and find Index, signature & Nature.

Sol:- Given Q.F is

$$3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3 \quad \text{—— (1)}$$

which is in the form of $a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{31}x_3x_1$ —— (2)

Comparing (1) & (2),

$a_{11}=3$; $a_{22}=3$; $a_{33}=3$; $2a_{12}=2$; $2a_{23}=-2$; $2a_{31}=2$

$\Rightarrow a_{12}=1$ $\quad \Rightarrow a_{23}=-1$ $\quad \Rightarrow a_{31}=1$

$\therefore$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Eigen values are $\lambda = 1, 4, 4$.

Eigen vector corresponding $\lambda=1$:—

$$(A-\lambda I)X=0$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\because \lambda=1)$$

(57)

$R_2 = 2R_2 - R_1$; $R_3 = 2R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by solving

$\Rightarrow 2x_1 + x_2 + x_3 = 0$; $3x_2 - 3x_3 = 0$; put $x_3 = k$

$\boxed{x_2 = x_3}$

$\Rightarrow 2x_1 = -x_2 - x_3$

$\Rightarrow \boxed{x_2 = k}$

$\Rightarrow 2x_1 = -k - k$

$2x_1 = -2k$

$\boxed{x_1 = -k}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

<u>Eigen vector corresponding</u> <u>$\lambda = 4$</u> :-

$$(A - \lambda I)X = 0$$

$$\Rightarrow (A - 4I)X = 0$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(\because \lambda = 4)$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$; $R_3 = R_3 + R_1$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1+x_2+x_3=0$; put $\boxed{\begin{matrix} x_3 = k_1 \\ x_2 = k_2 \end{matrix}}$

$-x_1 = -x_2 - x_3$

$\boxed{x_1 = k_1+k_2}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1+k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are eigen vectors.

But $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are not orthogonal to each other.

we can write the linear combination of above two vectors

$$a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} a+b \\ b \\ a \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ — (1)

i.e, $((a+b)\times 1)+(b\times 0)+(a\times 1)=0$

$\Rightarrow a+b+0+a=0$

$\Rightarrow 2a+b=0$

$\boxed{b=-2a}$

Sub. "b" value in (1)

$\Rightarrow \begin{bmatrix} a+b \\ b \\ a \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -a \\ -2a \\ a \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow a\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are the eigen vectors and orthogonal to eachother.

modal matrix $(P) = \left[\dfrac{x_1}{\|x_1\|}, \dfrac{x_2}{\|x_2\|}, \dfrac{x_3}{\|x_3\|}\right]$

$$P = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$P^T = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore PP^T = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$\therefore \boxed{PP^T = I} \Rightarrow \boxed{P^T = P^{-1}}$

Now $D = P^{-1}AP$

$= P^{T}AP$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Now. $Y^{T}DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ 4y_2 \\ 4y_3 \end{bmatrix}$

$$\boxed{Y^{T}DY = y_1^2 + 4y_2^2 + 4y_3^2}$$

Here Index $= 3 = s$

Signature $= 2s - r = 6 - 3 = 3.$

Nature $=$ +ve definite.

this is a Normal form (or) canonical form of given Q.F

"the orthogonal transformation which reduces the Q.F to canonical form is given by $\boxed{X = PY}$ $\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$" $\because$ 'P' is the matrix of the transformation

HW

(2). $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ convert into sum of squares form using orthogonal transformation.

(3). Reduce the Q.F $Q = 2(xy + yz + zx)$ to canonical form and find its rank, nature, index and signature by using orthogonal transformation.

Reduction of canonical form (or) Normal form (or) Sum of Squares form by using Diagonalisation (Linear transformation) :-

Step (1) :- For given Q.F into Symmetric matrix of order $n \times n$.

Step (2) :- Write $A_{n \times n} = I_n A I_n$ —— (1)

'I_n' is the Identity matrix of order 'n'.

Step (3) :- Apply row operations on L.H.S of eqⁿ (1) and apply the same operations in prefactor of 'A' on R.H.S of eqⁿ (1).

Step (4) :- Apply the column operations on L.H.S of eqⁿ (1) and apply the same operations on R.H.S of eqⁿ (1) on post factor of 'A'.

Step (5) :- Repeat the same procedure convert of the eqⁿ (1) into the form $\boxed{D = P^T A P}$, 'D' is the Diagonal matrix of order 'n'.

Step (6) :- the required canonical form is

$$Y^T D Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Step (7) :- the required linear transformation (or) matrix transformation is $\boxed{X = PY}$.

Problems

① Find the nature of the Q.F, Index, signature of Eqn $10x^2+2y^2+5z^2-4xy-10xz+6yz$ are reduced into canonical form by using Diagonalisation and find its transformation.

Sol:- Given Q.F is $10x^2+2y^2+5z^2-4xy-10xz+6yz$

which is in the form of $ax^2+by^2+cz^2+2hxy+2gzx+2fzy$

Here $a=10, b=2, c=5,\ 2h=-4,\ h=-2 \mid 2g=-10,\ g=-5 \mid 2f=6,\ f=3.$

$$\therefore A=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}=\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

We write $A_{3\times3}=I_3 A I_3$

$$\Rightarrow \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2=5R_2+R_1$; $R_3=2R_3+R_1$ to L.H.S and Prefactor of R.H.S

$$\Rightarrow \begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 4 & 5 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_3=2R_3-R_2$ to L.H.S and Prefactor of R.H.S.

$$\Rightarrow \begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $C_2 = 5C_2 + C_1$; $C_3 = 2C_3 + C_1$ to L.H.S and Post factor of R.H.S.

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 2 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Apply $C_3 = 2C_3 - C_2$ to L.H.S and Post factor of R.H.S

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 2 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

which is in the form of $\boxed{D = P^1 A P}$

where $D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ & $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$

∴ the required canonical form and Normal form is

$$Y^1 D Y = [y_1 \ y_2 \ y_3] \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\therefore \boxed{Y^1 D Y = 10y_1^2 + 40y_2^2} \quad \text{——} \quad (3)$$

Here Index $= S = 2$

Signature $= 2S - r = 4 - 2 = 2$ (or) Signature $= (+ve)$ terms $- (-ve)$ terms

$= 2 - 0$

$= 2.$

Nature = +ve semi definite

(In (3) having two positive terms and one term missing)

∴ the required matrix transformation and linear transformation is

$\boxed{X = PY}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x_1 = y_1$; $x_2 = y_1 + 5y_2 - 5y_3$; $x_3 = 4y_3$.

(2). Reduce the Q.f of $7x^2 + 6y^2 + 5z^2 - 4xy - 4yz$ to the canonical form by using Linear transformation.

(3). Reduce the Q.F to the canonical form of $2x^2 + 2y^2 + 2z^2 - 2xy + 2zx - 2yz$ by using orthogonal transformation.

(4). Reduce the Q.F $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_3x_2 + 4x_3x_1$ to the sum of the squares and find the corresponding Linear transformation. Find the index and signature.

(5). Reduce the Q.F to canonical form by an orthogonal reduction and state the nature of the Q.F

$5x^2 + 26y^2 + 10z^2 + 4yz + 14zx + 6xy$.

//

* Singular matrix :- A square matrix 'A' is said to be singular if $|A| = 0$ {if $|A| \neq 0$ non-singular matrix) 4

* Inverse of the matrix :- Let 'A' be any square matrix ∃ and 'B' ∋ $AB = BA = I$ then 'B' is called inverse of 'A', denoted by A^{-1}

* Adjoint of a square matrix :- Let 'A' be a square matrix of order 'n'. The transpose of the matrix got from 'A' by replacing the elements of 'A' by the corresponding co-factors is called the adjoint of 'A' and denoted by adj A.

* Theorems *

$\boxed{\text{Note}}$: For any scalar k, $adj(kA) = k^{n-1} adj(A)$

thm :- Every invertible matrix possesses a unique inverse.

(or)

the inverse of a matrix if it exists is unique.

Proof :- Let if possible, B and C be the inverses of 'A', then

$$AB = BA = I$$
&
$$AC = CA = I$$

Now
$$\begin{aligned} B &= BI \\ &= B(AC) \\ &= (BA)C \\ &= IC \\ B &= C \end{aligned}$$

Hence there is only one inverse of 'A', which is denoted by A^{-1}.

$\boxed{\text{Note}}$: $AA^{-1} = A^{-1}A = I$

& also $(A^{-1})^{-1} = A$ —— * $I^{-1} = I$

* also 'A' is invertible matrix and if $A = B$ then $A^{-1} = B^{-1}$

Note: $A^{-1} = \frac{\text{adj} A}{|A|}$, if $|A| \neq 0$ (2)

thm :- Every square matrix can be expressed as the sum of symmetric and skew-symmetric matrices in one and only way (uniquely).

(or)

Show that any square matrix $A = B + C$ where 'B' is symmetric & 'C' is skew-symmetric.

Proof :- Let 'A' be any square matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$
$$= P + Q \text{ (say)}$$

where $P = \frac{1}{2}(A + A^T)$, $Q = \frac{1}{2}(A - A^T)$

$$P^T = \left[\frac{1}{2}(A + A^T)\right]^T$$
$$= \frac{1}{2}(A + A^T)^T \quad [\because (kA)^T = kA^T]$$
$$= \frac{1}{2}(A^T + A) = P \Rightarrow \boxed{P^T = P}$$

$\therefore$ 'P' is a symmetric matrix.

& $Q^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A)$
$$= -\frac{1}{2}(A - A^T) = -Q \Rightarrow \boxed{Q^T = -Q}$$

$\therefore$ square matrix = symmetric + skew symmetric.

uniqueness :- If possible let $A = R + S$, where R = symmetric matrix, S = skew symmetric matrix

i.e. $R^T = R$ & $S^T = -S$

Now $A^T = (R + S)^T = R - S$ &

$\frac{1}{2}(A + A^T) = \frac{1}{2}(R + S + R - S) = R$

$\frac{1}{2}(A - A^T) = \frac{1}{2}(R + S - R + S) = S$

$\therefore R = P$ & $Q = S$

thus the representation is unique.

thm :- Prove that inverse of a non-singular symmetric matrix 'A' is symmetric. (9)

Proof :- Since 'A' is non-singular symmetric matrix.

$\therefore A^{-1}$ exists and $A^T = A$

Now, we have to prove that A^{-1} is symmetric.

we have $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ $\quad (\because A^T = A)$

Since $(A^{-1})^T = A^{-1} \Rightarrow A^{-1}$ is symmetric.

thm :- If 'A' is symmetric matrix, then prove that adj A is also symmetric.

Proof :- Since 'A' is symmetric, we have $A^T = A$

Now, we have $(\text{adj} A)^T = \text{adj} A^T$ $\quad [\because \text{adj} A^T = (\text{adj} A)^T]$

$= \text{adj} A$ $\quad (\because A^T = A)$

Since $(\text{adj} A)^T = \text{adj} A \Rightarrow \text{adj} A$ is a symmetric matrix.

thm :- If 'A' is a $m \times n$ matrix and 'B' is a $n \times p$ matrix then, $(AB)' = B'A'$

corollary :- $(ABC \ldots Z)' = Z'Y'X' \ldots C'B'A'$

thm :- If A, B are orthogonal matrices, each of order 'n' then AB and BA are orthogonal matrices.

Proof :- Since A and B are both orthogonal matrices

$AA^T = A^TA = I$ & $BB^T = B^TB = I$

Now $(AB)^T = B^TA^T$

$\Rightarrow (AB)^T(AB) = (B^TA^T)(AB) = B^T(A^TA)B = B^TIB$ $\quad (\because A^TA = AA^T)$

$= B^TB = I$ $\quad (\because BB^T = B^TB)$

$\Rightarrow (AB)$ is orthogonal. $\quad$ IIly BA is also orthogonal.

Thm :- Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

Proof :- Let 'A' be an orthogonal matrix

then $AA^T = A^TA = I$

Consider $AA^T = I$

Taking inverse on bothsides $\Rightarrow (AA^T)^{-1} = I^{-1}$

$\Rightarrow (A^T)^{-1} A^{-1} = I$

$\Rightarrow (A^{-1})^T A^{-1} = I$

$\therefore A^{-1}$ is orthogonal //

Again $A^TA = I$ $\Rightarrow$ Taking transpose on bothsides

$(A^TA)^T = I^T \Rightarrow A^T(A^T)^T = I \Rightarrow A^T$ is orthogonal //

$(\Rightarrow A^TA = I)$

Thm :- If A, B are invertible matrices of the same order, then

(i). $(AB)^{-1} = B^{-1}A^{-1}$ (ii). $(A^T)^{-1} = (A^{-1})^T$

Proof :- (i). we have $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

$\|^{ly}$ $(AB)(B^{-1}A^{-1}) = I$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

(ii). we have $AA^{-1} = A^{-1}A = I$

$\Rightarrow (AA^{-1})^T = (A^{-1}A)^T = I^T$

$\Rightarrow (A^{-1})^TA^T = A^T(A^{-1})^T = I$

$\Rightarrow (A^{-1})^T = (A^T)^{-1}$ ($\because$ by def. of inverse of the matrix)

Thm:- Every square matrix can be expressed as the sum of a symmetric and skew-symmetric matrices in one and only way (uniquely)

(or)

S.T any square matrix $A = B + C$ where 'B' is symmetric and 'C' is skew-symmetric matrices.

Proof:- Let 'A' be any square matrix. we can write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q \text{, say}$$

where $P = \frac{1}{2}(A + A^T)$ and $Q = \frac{1}{2}(A - A^T)$

we have $P^T = \left\{\frac{1}{2}(A + A^T)\right\}^T$

$$= \frac{1}{2}(A + A^T)^T \quad (\because (KA)^T = KA^T)$$

$$= \frac{1}{2}\left[A^T + (A^T)^T\right]$$

$$= \frac{1}{2}(A^T + A)$$

$$\therefore \boxed{P^T = P}$$

$\therefore$ 'P' is a symmetric matrix.

Now $Q^T = \left\{\frac{1}{2}(A - A^T)\right\}^T$

$$= \frac{1}{2}\left\{A^T - (A^T)^T\right\}$$

$$= \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}(A - A^T)$$

$$\therefore \boxed{Q^T = -Q}$$

$\therefore$ 'Q' is a skew-symmetric matrix.

They Square matrix = Symmetric matrix + skew symmetric matrix. ⑥

Hence the matrix 'A' is the sum of a symmetric matrix and a skew-symmetric matrix.

To prove that the sum is unique :-

If possible, let $A = R + S$

where 'R' is symmetric and 'S' is a skew-symmetric matrix.

$\therefore R^T = R$ & $S^T = -S$.

Now $A^T = (R+S)^T = R^T + S^T = R - S$ and

$\frac{1}{2}(A + A^T) = \frac{1}{2}(R + S + R - S) = R$

$\frac{1}{2}(A - A^T) = \frac{1}{2}(R + S - R + S) = S$.

$\Rightarrow R = P$ and $S = Q$.

$\therefore$ The representation is unique.

Hence Proved.

Thm :- S.T Every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Proof :- Let 'A' be square matrix.

Now $P = \frac{1}{2}(A + A^\theta)$, $Q = 1/2(A - A^\theta)$

we have $P^\theta = \left[\frac{1}{2}(A + A^\theta)\right]^\theta = \frac{1}{2}\left[A^\theta + (A^\theta)^\theta\right] = \frac{1}{2}(A^\theta + A) = P$.

$\therefore \boxed{P^\theta = P}$

$\therefore$ 'P' is Hermitian matrix.

(7).

Now $Q^{\theta} = \left\{\frac{1}{2}(A - A^{\theta})\right\}^{\theta} = \frac{1}{2}\left\{A^{\theta} - (A^{\theta})^{\theta}\right\} = \frac{1}{2}(A^{\theta} - A)$

$= -\frac{1}{2}(A - A^{\theta}) = -Q.$

$\therefore \boxed{Q^{\theta} = -Q}$

$\therefore$ 'Q' is a skew-Hermitian matrix.

$\therefore$ square matrix = Hermitian matrix + skew Hermitian matrix.

Hence the matrix 'A' is the sum of a Hermitian and a skew Hermitian matrix.

To prove that the representation is unique :-

Let $A = R + S$ be another such representation of 'A'.

where 'R' is Hermitian.

'S' is skew-Hermitian.

To prove that $R = P$ and $S = Q$.

then $A^{\theta} = (R+S)^{\theta} = R^{\theta} + S^{\theta} = R - S.$

$\therefore \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}(R+S+R-S) = R.$

$\frac{1}{2}(A - A^{\theta}) = \frac{1}{2}(R+S-R+S) = S.$

$\therefore R = P$ & $S = Q$.

$\therefore$ The representation is unique.

Hence proved.

Properties of Eigen values :-

2nd thm :- If 'λ' is Eigen value of an orthogonal matrix then $1/\lambda$ is also its eigen value.

Proof :- W.K.T if 'λ' is an eigen value of a matrix 'A', then '$1/\lambda$' is an eigen value of 'A^{-1}'.

Since 'A' is an orthogonal matrix, therefore $A^{-1} = A^T$.

$\therefore$ '$1/\lambda$' is an eigen value of 'A^T'.

But the matrices 'A' and 'A^T' have the same Eigen values, since the determinants $|A-\lambda I|$ and $|A^T-\lambda I|$ are same.

Hence '$1/\lambda$' is also an eigen value of 'A'.

Hence Proved.

Properties of eigen values :-

Theorem ①: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof :- i.e., If 'A' is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its n-eigen-values, then $\lambda_1 + \lambda_2 + \cdots + \lambda_n = Tr(A)$

& $\lambda_1 . \lambda_2 . \lambda_3 . \cdots \lambda_n = \det(A)$.

$\boxed{\text{Note}}$: $[|A - \lambda I_n| = (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-1} (\text{Trace } A) + \cdots = 0]$

characteristic equation of 'A' is $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Let 3×3 square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let 'λ' be the eigen value of 'A', then characteristic eqn $|A - \lambda I| = 0$

$\Rightarrow (-1)^3 \lambda^3 + \lambda^2 (\text{trace } A) + \cdots = 0$

Expanding this, we get

$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) - a_{12}(\text{a Polynomial of degree } n-2) +$

$\rightarrow |A - \lambda I| = -\lambda^3 + \lambda^2 (\text{tra } A) \ldots$

$a_{13}(\text{a Polynomial of degree } n-2) + \cdots + = 0$

$\lambda^3 = \lambda^2 (\text{tra } A)$

$\lambda = \text{tra } A$

i.e., $(-1)^n (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) \cdots (\lambda - a_{nn}) + \text{a polynomial of degree } (n-2) = 0$

i.e., $(-1)^n [\lambda^n - (a_{11} + a_{22} + \cdots + a_{nn}) \lambda^{n-1} + \text{a Polynomial of degree } (n-2)]$

$+ \text{a Polynomial of degree } (n-2) \text{ in } \lambda$

$\therefore (-1)^n \lambda^n + (-1)^{n+1} (\text{Trace } A) \lambda^{n-1} + \text{a Polynomial of degree } (n-2) \text{ in } \lambda = 0$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of this equation,

$$\text{Sum of the roots} = \frac{-(-1)^{n+1} Tr(A)}{(-1)^n} = Tr(A) \qquad \left(\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}\right)$$

Further $|A - \lambda I| = (-1)^n \lambda^n + \cdots + a_0$

Put $\lambda = 0 \Rightarrow |A| = a_0$

$(-1)^n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_0 = 0$

$\rightarrow$ Product of the roots $= \dfrac{(-1)^n a_0}{(-1)^n} = |A| = \det A$

Hence Proved (1).

Thm(2):- the Product of the eigen values of a matrix 'A' is equal to its determinant. (10)

Proof:- Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of $A_{n\times n}$, then

$\Rightarrow |A-\lambda I| = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n) = 0$

Put $\lambda = 0 \Rightarrow |A| = (-1)^n(-\lambda_1)(-\lambda_2)\ldots(-\lambda_n)$.

$= (-1)^n(-1)^n \lambda_1\cdot\lambda_2\ldots\lambda_n$

$= (-1)^{2n}\lambda_1\cdot\lambda_2\ldots\lambda_n$

$\therefore |A| = \lambda_1\cdot\lambda_2\ldots\lambda_n$

$\therefore$ the Product of the eigen values of a matrix 'A' is equal to its determinant.

Thm(3):- the Sum of the eigen values of a matrix is the trace of the matrix.

Proof:- Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

the characteristic equation is $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$

$\Rightarrow (a_{11}-\lambda)\{(a_{22}-\lambda)(a_{33}-\lambda) - a_{23}a_{32}\} - a_{12}\{a_{21}(a_{33}-\lambda) - a_{31}a_{23}\} + a_{13}\{a_{21}a_{32} - a_{31}(a_{22}-\lambda)\} = 0$

$\Rightarrow (a_{11}-\lambda)\{a_{22}a_{33} - a_{22}\lambda - a_{33}\lambda + \lambda^2 \ldots$

$\Rightarrow a_{11}a_{22}a_{33} - a_{11}a_{22}\lambda - a_{11}a_{33}\lambda + a_{11}\ldots - a_{22}a_{33}\lambda + \lambda^2 a_{22} + \lambda^2 a_{33} - \lambda^3 \ldots = 0$

$\Rightarrow -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) + \ldots = 0$

$\Rightarrow |A-\lambda I| = -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) + \ldots$ ——— (1)

Also, if $\lambda_1, \lambda_2, \lambda_3$ be the eigen value then

$\Rightarrow |A-\lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$

$\Rightarrow |A-\lambda I| = -\lambda^3 + \lambda^2(\lambda_1+\lambda_2+\lambda_3) + \ldots$ ——— (2)

thm (4) :- If 'λ' is an eigen value of 'A' corresponding to the eigen vector (1) 'x', then λ^n is eigen value of A^n, corresponding to the eigen vector 'x';

Proof :- Since 'λ' is eigen value of 'A' corresponding to the eigen-vector 'x', we have $\boxed{Ax = \lambda x}$ ——— (1) by using mathematical Induction

Premultiply (1) by A, $A(Ax) = A(\lambda x)$

i.e., $(AA)x = \lambda(Ax) \rightarrow \boxed{A^2x = \lambda^2 x}$ ——— (2) [$\because$ by (1)]

Hence 'λ^2' is eigen value of 'A^2' with 'x' itself as the corresponding eigen vector. thus the theorem is true to $n = 2$.

Let the result be true for $n = k$

then $A^k x = \lambda^k x$

Pre multiplying this by (A) and using $Ax = \lambda x$, we get $A^{k+1} x = \lambda^{k+1} x$

$\Rightarrow \lambda^{k+1}$ is eigen value of A^{k+1} with 'x' itself as the corresponding eigen vector.

Hence by the Principle of mathematical induction, the thm is true for all Positive integers 'n'.

thm (5) :- S.T if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the latent roots of A, then A^3 has latent roots $\lambda_1^3, \lambda_2^3, \ldots, \lambda_n^3$.

Proof :- Since 'λ' is eigen value of 'A' corresponding to the eigen vector 'x', we have $\boxed{Ax = \lambda x}$ ——— (1) By using mathematical induction

Premultiplying (1) by 'A', $A(Ax) = A(\lambda x)$

i.e., $\boxed{A^2 x = \lambda_i^2 x}$ ——— (2)

Again Premultiplying (2) by 'A' $\rightarrow A(A^2x) = A(\lambda^2 x)$

$\Rightarrow$ Hence this is true for all $\rightarrow \boxed{A^3 x = \lambda_i^3 x}$ ——— (3)

$i = 1, 2, 3, \ldots$

Thm⑥:- A square matrix 'A' and its transpose A^T have the same eigen values.

Proof:- The characteristic matrix of 'A' is $(A-\lambda I)$

& the characteristic matrix of 'A^T' is $(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$\therefore |(A-\lambda I)^T| = |A^T - \lambda I|$ (or) $|A^T - \lambda I| = |A - \lambda I|$ $(\because |A^T| = |A|)$

$\therefore |A - \lambda I| = 0 \Leftrightarrow |A^T - \lambda I| = 0$

i.e, 'λ' is an eigen value of 'A' $\Leftrightarrow$ 'λ' is an eigen value of A^T.

thus the eigen values of 'A' & 'A^T' are same.

Hence proved.

Def:- Two matrices 'A' & 'B' are said to be "similar", if $\exists$ an invertible matrix 'P' $\ni$ $\boxed{B = P^{-1}AP}$

Thm⑦:- If 'A' and 'B' are n-rowed square matrices and if 'A' is invertible. show that $A^{-1}B$ and BA^{-1} have same eigen values.

Proof:- Given 'A' is invertible $\Rightarrow$ 'A^{-1}' exists.

Now $A^{-1}B = A^{-1}BI$

$= A^{-1}B(A^{-1}A)$ $\quad (\because A^{-1}A = I)$

$= A^{-1}(BA^{-1})A$

$A^{-1}B = A^{-1}(BA^{-1})A$ ——— (1)

Since Eigen values of two similar matrices are same, so matrices BA^{-1} and $A^{-1}(BA^{-1})A$ have the same eigen values, so, by (1) the matrices $A^{-1}B$ and BA^{-1} have the same eigen values.

Hence proved.

Theorem⑧:- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of A, then $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the eigen values of kA. Where 'k' is a non-zero scalar.

Proof :- Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of 'A'.

Case (1) :- $k=0$, then $kA = O$ and each eigen value of 'O' is 0.

then $0\lambda_1, 0\lambda_2, \ldots 0\lambda_n$ are the eigen values of kA.

Case (2) :- $k \neq 0 \Rightarrow |kA - \lambda k I| = |k(A - \lambda I)|$

$= k^n |A - \lambda I|$ $\quad [\because |kB| = k^n |B|]$

If $k \neq 0$, then $|kA - \lambda k I| = 0 \Leftrightarrow |A - \lambda I| = 0$

this shows that '$k\lambda$' is eigen value of 'kA' $\Leftrightarrow$ 'λ' is eigen value of 'A'

Hence $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigen values of kA if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of 'A'. 'k' is a non-zero scalar.

Hence Proved.

Imp

thm⑨:- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of 'A', then $\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k$ are the eigen values of the matrix $(A - kI)$, where 'k' is a non-zero scalar.

Proof :- Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of 'A'.

the characteristic polynomial of 'A' is $|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$ — (1)

thus the characteristic polynomial of $A - kI$ is

$$|(A - kI - \lambda I)| = |A - (k+\lambda) I|$$

$$= [\lambda_1 - (\lambda + k)][\lambda_2 - (\lambda + k)] \ldots [\lambda_n - (\lambda + k)] \quad [\because \text{from (1)}]$$

$$= [(\lambda_1 - k) - \lambda][(\lambda_2 - k) - \lambda] \ldots [(\lambda_n - k) - \lambda]$$

this shows that the eigen values of $A - kI$ are $\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k$

Thm (10) :- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, Find the eigen-values of the matrix $(A-\lambda I)^2$. (104)

Proof :- First we will find the eigen values of the matrix $A-\lambda I$.

Since, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of 'A'.

$\therefore$ the characteristic polynamial of 'A' is $|A-kI| = (\lambda_1-k)(\lambda_2-k)\ldots(\lambda_n-k)$
where 'k' is a scalar. —— (1)

The characteristic polynamial of the matrix $(A-\lambda I)$ is

$|A-\lambda I-kI| = |A-(\lambda+k)I|$

$= [\lambda_1-(\lambda+k)][\lambda_2-(\lambda+k)]\ldots[\lambda_n-(\lambda+k)]$ [$\because$ from (1)]

$= [(\lambda_1-\lambda)-k][(\lambda_2-\lambda)-k]\ldots[(\lambda_n-\lambda)-k]$

Which shows that the eigen values of $A-\lambda I$ are $\lambda_1-\lambda, \lambda_2-\lambda, \ldots, \lambda_n-\lambda$

We know that, if the eigen values of 'A' are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigen values of 'A^2' are $\lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots \lambda_n^2$.

thus the eigen values of $(A-\lambda I)^2$ are $(\lambda_1-\lambda)^2, (\lambda_2-\lambda)^2, \ldots (\lambda_n-\lambda)^2$.

Hence Proved.

Thm (11) / If 'λ' is an eigen value of a non-singular matrix 'A', corresponding to the eigen vector 'x', then 'λ^{-1}' is an eigen value of A^{-1} and corresponding eigen vector 'x' itself. (or) If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the Eigen values of 'A' then (or) $1/\lambda_1, 1/\lambda_2, \ldots 1/\lambda_n$ are the eigen values of A^{-1}

Prove that the eigen values of 'A^{-1}' are the reciprocals of the eigen values of 'A'

Proof :- Since 'A' is non-singular & product of the eigen values is equal to $|A|$, it follows that none of the eigen values of 'A' is '0'.

$\therefore$ If 'λ' is an eigen value of the non-singular matrix 'A' and 'x' is corresponding eigen vector, $\lambda \neq 0$ and $\boxed{Ax_i = \lambda x_i}$

Premultiplying 'A^{-1}', we get

$$\Rightarrow A^{-1}(Ax_i) = A^{-1}(\lambda x_i) \Rightarrow (A^{-1}A)x = \lambda A^{-1}x$$

$$\Rightarrow x_i = A^{-1}(\lambda x_i) \qquad \Rightarrow Ix = \lambda A^{-1}x$$

$$\boxed{\frac{1}{\lambda}x_i = A^{-1}x_i} \qquad \therefore x = \lambda A^{-1}x$$

$$\Rightarrow \boxed{A^{-1}x = \lambda^{-1}x} \quad (\because \lambda \neq 0)$$

Hence by definition it follows that 'λ^{-1}' is an eigen value of 'A^{-1}' and 'x' is the corresponding eigen vector.

Hence Proved.

Thm (12):- If 'λ' is an eigen value of a non-singular matrix 'A', then $\frac{|A|}{\lambda}$ is an eigen value of the matrix $\operatorname{adj}A$.

Proof:- Since 'λ' is an eigen value of a non-singular matrix, therefore, $\lambda \neq 0$.

Also 'λ' is an eigen value of 'A' $\Rightarrow \exists$ a non zero vector 'x' $\ni$

$$\boxed{Ax = \lambda x} \quad \text{———} \quad (1)$$

$$\Rightarrow (\operatorname{adj}A)(Ax) = (\operatorname{adj}A)(\lambda x) \Rightarrow [(\operatorname{adj}A)A]x = \lambda(\operatorname{adj}A)x$$

$$\Rightarrow |A|Ix = \lambda(\operatorname{adj}A)x \qquad [\because (\operatorname{adj}A)A = |A|I]$$

$$\Rightarrow \frac{|A|}{\lambda}x = (\operatorname{adj}A)x$$

$$\Rightarrow (\operatorname{adj}A)x = \frac{|A|}{\lambda}x$$

Since 'x' is a non-zero vector, therefore, from the relation (1), it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix $\operatorname{adj}A$.

Hence Proved.

thm(13):- If 'λ' is an eigen value of 'A', then Prove that the eigen value of $B = a_0A^2 + a_1A + a_2I$ is $a_0\lambda^2 + a_1\lambda + a_2$.

Proof:- If 'x' be the eigen vector corresponding to the eigen value 'λ', then

$$\boxed{AX = \lambda X} \quad \text{——— (1)}$$

Premultiply by 'A' on both sides, $\Rightarrow A(AX) = A(\lambda X)$

$$\Rightarrow A^2X = \lambda(AX)$$

$$\Rightarrow \boxed{A^2X = \lambda^2 X}$$

this shows that 'λ^2' is an eigen value of 'A^2'.

We have $B = a_0A^2 + a_1A + a_2I$

$$\therefore BX = (a_0A^2 + a_1A + a_2I)X = a_0A^2X + a_1AX + a_2X$$

$$= a_0\lambda^2X + a_1\lambda X + a_2X$$

$$= (a_0\lambda^2 + a_1\lambda + a_2)X$$

this shows that $a_0\lambda^2 + a_1\lambda + a_2$ is an eigen value of 'B' and the corresponding eigen vector of 'B' is 'x'.

Hence Proved.

thm(14):- Suppose that 'A' and 'P' are square matrices of order 'n' $\ni$ P is non-singular then 'A' and $P^{-1}AP$ have the same eigen values.

Proof:- Consider the characteristic equation of $P^{-1}AP$ is

$$|(P^{-1}AP) - \lambda I| = |P^{-1}AP - \lambda P^{-1}IP| \quad (\because I = P^{-1}IP)$$

$$= |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P|$$

$$= |A - \lambda I| \quad (\because |P^{-1}||P| = 1)$$

thus the characteristic polynomials of '$P^{-1}AP$' and 'A' are same.

Hence the eigen values of '$P^{-1}AP$' & 'A' are same.

Hence Proved.

multiplying both sides of (2) by 'A', we get (19)

$$A(k_1 x_1 + k_2 x_2) = A(0) = 0.$$

$$\Rightarrow k_1 (A x_1) + k_2 (A x_2) = 0$$

$$\Rightarrow k_1 (\lambda_1 x_1) + k_2 (\lambda_2 x_2) = 0 \quad \text{(3)} \quad [\because (1)]$$

(3) $- \lambda_2$(2) gives

$$\Rightarrow k_1(\lambda_1 x_1) + k_2(\lambda_2 x_2) - \lambda_2[k_1 x_1 + k_2 x_2] = 0$$

$$\Rightarrow k_1(\lambda_1 x_1) + \cancel{k_2(\lambda_2 x_2)} - k_1\lambda_2 x_1 - \cancel{k_2 \lambda_2 x_2} = 0$$

$$\Rightarrow k_1(\lambda_1 - \lambda_2)x_1 = 0$$

$$\Rightarrow k_1 = 0 \quad (\because \lambda_1 \neq \lambda_2) \text{ \& } x_1 \neq 0.$$

$$\Rightarrow k_2 = 0 \quad (\text{lly we get})$$

But this contradicts our assumption that k_1, k_2 are not zeros.

Hence our assumption that 'x_1' and 'x_2' are linearly dependent is wrong.

Hence the two eigen vectors corresponding to the two different eigen values are linearly independent (L.I)

Hence Proved.